BESICOVITCH-FEDERER PROJECTION THEOREM AND GEODESIC FLOWS ON RIEMANN SURFACES

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ABSTRACT. We extend the Besicovitch-Federer projection theorem to transversal families of mappings. As an application we show that on a certain class of Riemann surfaces with constant negative curvature and with boundary, there exist natural 2-dimensional measures invariant under the geodesic flow having 2-dimensional supports such that their projections to the base manifold are 2-dimensional but the supports of the projections are Lebesgue negligible. In particular, the union of complete geodesics has Hausdorff dimension 2 and is Lebesgue negligible.

1. INTRODUCTION

A *pair of pants* S is a 2-sphere minus three points endowed with a metric of constant curvature -1 in such a way that the boundary consists of three closed geodesics of length a, b and c called the *cuffs*. The metric is uniquely determined by these three lengths. (For more details, see e.g. [H].) For each point x in S, write Ω_x for the set of unit tangent vectors $v \in T^1_x S$ such that the geodesic ray $\gamma_v(t), t \geq 0$, with initial condition (x, v) never meets the boundary ∂S of S. The set Ω_x is a Cantor set of dimension $\delta = \delta(a, b, c)$. The number δ is an important geometric invariant of the pair of pants S: it is the critical exponent of the Poincaré series of $\pi_1(S)$ and the topological entropy of the geodesic flow on $T^1 S$ (cf. [S2]). We will recall in Section 3 why the function $(a, b, c) \mapsto \delta$ is real analytic. In particular, the function $a \mapsto \delta(a, a, a)$ is continuous from $(0, \infty)$ onto the open set $(0, 1)$. In a very similar setting, McMullen ([Mc]) gives asymptotics for $1 - \delta(a, a, a)$ when $a \to 0$ and for $\delta(a, a, a)$ when $a \to \infty$.

We are interested in the set

$$C(S) := \{x \in S \mid \text{there exists } v \in T^1_x S \text{ such that } v \in \Omega_x \text{ and } -v \in \Omega_x\}. \tag{1.1}$$

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In other words, $C(S)$ is the set of points in complete geodesics in S. Let

$$D(S) := \{(x,v) \in T^1S \mid x \in C(S), v \in \Omega_x, -v \in \Omega_x\} \tag{1.2}$$

be the subset of T^1S where the geodesic flow is defined for all $t \in \mathbb{R}$. Clearly, $\Pi(D(S)) = C(S)$, where $\Pi : T^1S \to S$, $\Pi(x,v) = x$, is the canonical projection.

We write $\mathcal{L}^l$ and $\mathcal{H}^s$ to denote the l-dimensional Lebesgue measure and the s-dimensional Hausdorff measure. For the Hausdorff dimension we use the notation $\dim_{\mathrm{H}}$.

We consider the following theorem:

Theorem 1.1. *With the above notation,*

- $\mathcal{L}^2(C(S)) > 0$ *provided that* $\delta > 1/2$ *and*
- $\dim_{\mathrm{H}} C(S) = 1 + 2\delta$ *and* $\mathcal{L}^2(C(S)) = 0$ *provided that* $\delta \leq 1/2$.

It is known that $\dim_{\mathrm{H}}(D(S)) = 1 + 2\delta$ (see Section 3). Ledrappier and Lindenstrauss proved in [LL] (see [JJL] for a different proof) that Π does not diminish the Hausdorff dimension of a measure which is invariant under the geodesic flow. The new part of our result is when δ is exactly $1/2$. In that case, [LL] implies that $\dim_{\mathrm{H}} C(S) = 2$, and we sharpen this by proving that $C(S)$ is Lebesgue negligible.

The main technical part of our paper is the following extension of Besicovitch-Federer projection theorem to transversal families of maps. (For the definition of transversality, see Definition 2.4.) We believe that Theorem 1.2 is of independent interest (see for example [OS]), and therefore we verify it in a more general setting than needed for the purpose of proving Theorem 1.1.

Theorem 1.2. *Let* $E \subset \mathbb{R}^n$ *be* $\mathcal{H}^m$*-measurable with* $\mathcal{H}^m(E) < \infty$. *Assume that* $\Lambda \subset \mathbb{R}^l$ *is open and* $\{P_\lambda : \mathbb{R}^n \to \mathbb{R}^m\}_{\lambda \in \Lambda}$ *is a transversal family of maps. Then* E *is purely* m*-unrectifiable, if and only if* $\mathcal{H}^m(P_\lambda(E)) = 0$ *for* $\mathcal{L}^l$*-almost all* $\lambda \in \Lambda$.

In [HJJL] we showed that on any Riemann surface with (variable) negative curvature there exist 2-dimensional measures which are invariant under the geodesic flow and have singular projections with respect to $\mathcal{L}^2$. The measures are supported by the whole unit tangent bundle T^1S and they are singular with respect to $\mathcal{H}^2$ on T^1S. However, the measures constructed in this paper have 2-dimensional supports and they are absolutely continuous with respect to $\mathcal{H}^2$ on T^1S. Thus their singularity is due to the projection.

The paper is organized as follows: In Section 2 we introduce the notation and prove Theorem 1.2. In Section 3 we recall basic properties of the geodesic flow on a pair of pants and prove Theorem 1.1 as an application of Theorem 1.2.

2. PROJECTIONS

In this section we prove Theorem 1.2 as a consequence of several lemmas. In the case of orthogonal projections in $\mathbb{R}^n$, one can find a proof for the "only if"-part of Theorem 1.2 in [Ma, Chapter 18] or in [F, Chapter 3.3]. The main idea of our proof is same as that of [Ma], but, due to our more general setting, some

modifications are naturally needed – the major ones being in Lemma 2.5. For the convenience of the reader we give the main arguments. In fact, our approach simplifies slightly the corresponding arguments in [Ma].

In this section $\Lambda \subset \mathbb{R}^l$ is open and l, m and n are integers with $m \leq l$ and $m < n$. The closed ball with radius r centred at x is denoted by $B(x,r)$. As in [Ma], a non-negative, subadditive set function vanishing for the empty set is called a measure. We start by defining cones around preimages of points with respect to Lipschitz continuous mappings.

Definition 2.1. Let $\lambda \in \Lambda$ and let $P_\lambda : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz continuous. For all $a \in \mathbb{R}^n$, $0 < s < 1$ and $r > 0$, we define

$$X(a,\lambda,s) := \{x \in \mathbb{R}^n \mid |P_\lambda(x) - P_\lambda(a)| < s|x-a|\} \text{ and}$$
$$X(a,r,\lambda,s) := X(a,\lambda,s) \cap B(a,r).$$

The following lemma is an analogue of [Ma, Corollary 15.15].

Lemma 2.2. *Suppose that $E \subset \mathbb{R}^n$ is purely m-unrectifiable. Let $\delta > 0$ and $\lambda \in \Lambda$. Defining*

$$E_{1,\delta}(\lambda) := \{a \in E \mid \limsup_{s\to 0} \sup_{0<r<\delta} (rs)^{-m}\mathcal{H}^m(E \cap X(a,r,\lambda,s)) = 0\},$$

we have $\mathcal{H}^m(E_{1,\delta}(\lambda)) = 0$.

Proof. Replacing Q_V by P_λ in [Ma, Lemmas 15.13 and 15.14] and observing that the Lipschitz constant of Q_V is one, the proof of [Ma, Corollary 15.15] works in our setting. Here Q_V is the projection onto the orthogonal complement $V^\perp$ of an m-plane going through the origin. □

Next we consider the analogue of [Ma, Lemmas 18.3 and 18.4] in our setting. The proof of [Ma, Lemma 18.3] relies on the fact that $Q_V(\{x \in B(a,r) \mid |Q_V(x-a)| < s|x-a|\}) = U(Q_V(a), rs) \cap V^\perp$ where $U(z,r)$ is the open ball with centre at z and with radius r. Note that this does not hold when Q_V is replaced by P_λ. However, the proof given in [F, Lemma 3.3.9] works in our setting.

Lemma 2.3. *Let $E \subset \mathbb{R}^n$ with $\mathcal{H}^m(E) < \infty$, $\delta > 0$ and $\lambda \in \Lambda$. Defining*

$$E_{2,\delta}(\lambda) := \{a \in E \mid \limsup_{s\to 0} \sup_{0<r<\delta} (rs)^{-m}\mathcal{H}^m(E \cap X(a,r,\lambda,s)) = \infty\}$$

and

$$E_3(\lambda) := \{a \in E \mid \#(E \cap P_\lambda^{-1}(P_\lambda(a))) = \infty\},$$

we have $\mathcal{H}^m(P_\lambda(E_{2,\delta}(\lambda))) = 0$ and $\mathcal{H}^m(P_\lambda(E_3(\lambda))) = 0$.

Proof. The first claim can be verified in the same way as [F, Lemma 3.3.9] and the latter one follows from [Ma, Theorem 7.7]. □

Throughout the rest of this section we assume that the family $\{P_\lambda : \mathbb{R}^n \to \mathbb{R}^m\}_{\lambda\in\Lambda}$ is transversal. We use a slight variant of the $\beta = 0$ case of the definition of β-transversality given in [PS, Definition 7.2].

Definition 2.4. Let $\Lambda \subset \mathbb{R}^l$ be open. A family of maps $\{P_\lambda : \mathbb{R}^n \to \mathbb{R}^m\}_{\lambda \in \Lambda}$ is transversal if it satisfies the following conditions for each compact set $K \subset \mathbb{R}^n$:

(1) The mapping $P : \Lambda \times K \to \mathbb{R}^m$, $(\lambda, x) \mapsto P_\lambda(x)$, is continuously differentiable and twice differentiable with respect to λ.

(2) For $j = 1, 2$ there exist constants C_j such that the derivatives with respect to λ satisfy

$$\|D^j_\lambda P(\lambda, x)\| \leq C_j \text{ for all } (\lambda, x) \in \Lambda \times K.$$

(3) For all $\lambda \in \Lambda$ and $x, y \in K$ with $x \neq y$, define

$$T_{x,y}(\lambda) := \frac{P_\lambda(x) - P_\lambda(y)}{|x - y|}.$$

Then there exists a constant $C_T > 0$ such that the property

$$|T_{x,y}(\lambda)| \leq C_T$$

implies that

$$\det\Big(D_\lambda T_{x,y}(\lambda)\,(D_\lambda T_{x,y}(\lambda))^T\Big) \geq C_T^2.$$

(4) There exists a constant C_L such that

$$\|D^2_\lambda T_{x,y}(\lambda)\| \leq C_L$$

for all $\lambda \in \Lambda$ and $x, y \in K$ with $x \neq y$.

Next we verify the analogue of [Ma, Lemma 18.9].

Lemma 2.5. *Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$-measurable with $\mathcal{H}^m(E) < \infty$ and let $\delta > 0$. For $\mathcal{L}^l$-almost all $\lambda \in \Lambda$ we have for $\mathcal{H}^m$-almost all $a \in E$ either*

$$\limsup_{s \to 0} \sup_{0<r<\delta} (rs)^{-m}\mathcal{H}^m(E \cap X(a, r, \lambda, s)) = 0 \text{ or} \tag{2.1}$$

$$\limsup_{s \to 0} \sup_{0<r<\delta} (rs)^{-m}\mathcal{H}^m(E \cap X(a, r, \lambda, s)) = \infty \text{ or} \tag{2.2}$$

$$(E \setminus \{a\}) \cap P_\lambda^{-1}(P_\lambda(a)) \cap B(a, \delta) \neq \emptyset. \tag{2.3}$$

Proof. Given $\delta > 0$ and $a \in E$, we prove that for $\mathcal{L}^l$-almost all $\lambda \in \Lambda$ either (2.1), (2.2) or (2.3) holds. Then the claim follows by Fubini's theorem. The measurability arguments needed for applying Fubini's theorem are similar as those in [F, Lemma 3.3.2]. We may clearly suppose that $E \subset K$ for some compact $K \subset \mathbb{R}^n$, and furthermore, by [Ma, Theorem 1.10] E may be assumed to be σ-compact.

Fix $a \in E$, $\lambda_0 \in \Lambda$ and $0 < \delta < \delta_0$ such that $B(\lambda_0, 2\delta_0) \subset \Lambda$. Let $V \subset \mathbb{R}^l$ be an m-dimensional linear subspace and let $V_{\lambda_1} = V + \lambda_1$ for all $\lambda_1 \in \Lambda$. For all $\lambda_1 \in B(\lambda_0, \delta_0)$, define a measure $\Psi_{V_{\lambda_1}}$ on $B(\lambda_0, 2\delta_0) \cap V_{\lambda_1}$ by

$$\Psi_{V_{\lambda_1}}(A) := \sup_{0<r<\delta} r^{-m}\mathcal{H}^m(E \cap B(a, r) \cap L_{V_{\lambda_1}}(A))$$

for all $A \subset B(\lambda_0, 2\delta_0) \cap V_{\lambda_1}$, where

$$L_{V_{\lambda_1}}(A) := \bigcup_{\lambda \in A} P_\lambda^{-1}(P_\lambda(a)).$$

The set

$$C_{V_{\lambda_1}} := \{\lambda \in B(\lambda_0, 2\delta_0) \cap V_{\lambda_1} \mid (E \setminus \{a\}) \cap L_{V_{\lambda_1}}(\{\lambda\}) \cap B(a, \delta) \neq \emptyset\}$$

is $\mathcal{H}^m$-measurable. This follows from the fact that it is σ-compact which can be seen as follows: defining a continuous function

$$\widetilde{P} : (B(\lambda_0, 2\delta_0) \cap V_{\lambda_1}) \times \mathbb{R}^n \to \mathbb{R}^m, \ \widetilde{P}(\lambda, x) := P_\lambda(x) - P_\lambda(a),$$

and σ-compact sets

$$S_1 := \{(\lambda, x) \in (B(\lambda_0, 2\delta_0) \cap V_{\lambda_1}) \times \mathbb{R}^n \mid \widetilde{P}(\lambda, x) = 0\}$$

and

$$S_2 := S_1 \cap (B(\lambda_0, 2\delta_0) \times ((E \setminus \{a\}) \cap B(a, \delta))),$$

we conclude that $C_{V_{\lambda_1}} = \Pi_\Lambda(S_2)$, where $\Pi_\Lambda : \Lambda \times \mathbb{R}^n \to \Lambda$ is the projection $\Pi_\Lambda(\lambda, x) = \lambda$. Thus $C_{V_{\lambda_1}}$ is σ-compact.

Let $D_{V_{\lambda_1}} := (B(\lambda_0, 2\delta_0) \cap V_{\lambda_1}) \setminus C_{V_{\lambda_1}}$. From the definitions of $\Psi_{V_{\lambda_1}}$ and $C_{V_{\lambda_1}}$ we deduce that $\Psi_{V_{\lambda_1}}(D_{V_{\lambda_1}}) = 0$. Now [Ma, Theorem 18.5] implies that for $\mathcal{H}^m$-almost all $\lambda \in B(\lambda_0, \delta_0) \cap V_{\lambda_1}$ either

$$\limsup_{t \downarrow 0} t^{-m} \Psi_{V_{\lambda_1}} (B(\lambda_0, 2\delta_0) \cap V_{\lambda_1} \cap B(\lambda, t)) = 0 \tag{2.4}$$

or

$$\limsup_{t \downarrow 0} t^{-m} \Psi_{V_{\lambda_1}} (B(\lambda_0, 2\delta_0) \cap V_{\lambda_1} \cap B(\lambda, t)) = \infty \tag{2.5}$$

or

$$\lambda \in C_{V_{\lambda_1}}. \tag{2.6}$$

Applying Fubini's theorem we see that for $\mathcal{L}^l$-almost all $\lambda \in B(\lambda_0, \delta_0)$ either (2.4), (2.5) or (2.6) holds with V_{λ_1} replaced by V_λ. (The measurability proofs needed here can be dealt with in a similar manner as those in [F, Lemma 3.3.3].) Note that here the exceptional set of $\mathcal{L}^l$-measure zero depends on the m-plane V. Hence it is sufficient to find a finite collection of linear m-planes $V^1, \dots, V^k \subset \mathbb{R}^l$ and $C > 0$ such that for all $\lambda \in B(\lambda_0, \delta_0)$

$$\bigcup_{j=1}^{k} B(a, r) \cap L_{V_\lambda^j}(B(\lambda_0, 2\delta_0) \cap V_\lambda^j \cap B(\lambda, C^{-1}s)) \setminus \{a\} \subset X(a, r, \lambda, s)$$
$$\subset \bigcup_{j=1}^{k} B(a, r) \cap L_{V_\lambda^j}(B(\lambda_0, 2\delta_0) \cap V_\lambda^j \cap B(\lambda, Cs)) \setminus \{a\}$$

for every small enough $s > 0$. Indeed, by [JJN, Lemma 3.3] there are $C > 0$ and $s_0 > 0$ such that for any $0 < s < s_0$ and for any $x \in X(a, r, \lambda, s)$ there exists an m-dimensional coordinate plane W such that $x \in L_{W_\lambda}(B(\lambda_0, 2\delta_0) \cap W_\lambda \cap$

$B(\lambda, Cs))$, giving the latter inclusion for the collection of all m-dimensional coordinate planes in $\mathbb{R}^l$. Finally, the first inclusion is true for any m-plane since, by transversality, $\|D_\lambda T_{x,a}(\lambda)\|$ is bounded. □

For the "if"-part of Theorem 1.2 we need the following lemma.

Lemma 2.6. *Assume that $\{P_\lambda : \mathbb{R}^n \to \mathbb{R}^m\}_{\lambda\in\Lambda}$ is a transversal family of mappings. Then for every $a \in \mathbb{R}^n$, for every m-dimensional C^1-submanifold $S \subset \mathbb{R}^n$ containing a and for $\mathcal{L}^l$-almost all $\lambda \in \Lambda$ there exist $\gamma > 0$ and $r > 0$ such that $|P_\lambda(x) - P_\lambda(y)| \geq \gamma|x - y|$ for all $x, y \in B(a, r) \cap S$.*

Proof. We begin by showing that P_λ is a submersion, that is, $D_xP_\lambda(a)$ has rank m at every point $a \in \mathbb{R}^n$. Here D_xP_λ is the derivative of P_λ with respect to x.

Let $\lambda_0 \in \Lambda$ and let $\ker D_xP_\lambda(a) \subset \mathbb{R}^n$ be the kernel of $D_xP_\lambda(a)$. By [JJN, Lemma 3.2], Definition 2.4 implies that for any unit vector $e \in \ker D_xP_{\lambda_0}(a)$ one can find an m-dimensional plane $V^e \subset \mathbb{R}^l$ such that the mapping $g^e : V^e_{\lambda_0} \cap \Lambda \to \mathbb{R}^m$, defined as $g^e(\lambda) := D_xP_\lambda(a)(e)$, is a diffeomorphism (onto its image) on a small neighbourhood of λ_0. Furthermore, the parallelepiped $Dg^e(\lambda_0)([-1, 1]^m)$ is uniformly thick – by this we mean that the lengths of the edges and the angles between the edges are bounded from below by a constant which is independent of $\lambda_0 \in \Lambda$, $e \in \ker D_xP_{\lambda_0}(a)$ and $a \in K$ for any fixed compact $K \subset \mathbb{R}^n$.

Since $D_xP_\lambda(a)$ is continuous in λ and $\dim\ker D_xP_\lambda(a) \geq n - m$ for all $\lambda \in \Lambda$ there is $e \in \ker D_xP_{\lambda_0}(a)$ such that $e = \lim_{\lambda\to\lambda_0} e_\lambda$, where $e_\lambda \in \ker D_xP_\lambda(a)$. Define a function $f^e : V^e_{\lambda_0} \cap \Lambda \to \mathbb{R}^n$ by

$$f^e(\lambda) := e - \mathrm{proj}_{\ker D_xP_\lambda(a)}(e),$$

where proj_V is the orthogonal projection onto $V \subset \mathbb{R}^n$. Observe that $g^e(\lambda) = D_xP_\lambda(a)(f^e(\lambda))$. The fact that $Dg^e(\lambda_0)([-1, 1]^m)$ is uniformly thick implies that the same is true for $Df^e(\lambda_0)([-1, 1]^m)$.

Assuming that $\dim\ker D_xP_{\lambda_0}(a) > n - m$ there are at most $m - 1$ directions perpendicular to $\ker D_xP_{\lambda_0}(a)$. Thus $Df^e(\lambda_0)([-1, 1]^m)$ intersects $\ker D_xP_{\lambda_0}(a)$ in a set containing a line segment of positive length. In particular, there is a unit vector $v \in V^e$ satisfying $Df^e(\lambda_0)(v) \in \ker D_xP_{\lambda_0}(a)$ which, in turn, gives the contradiction $Dg^e(\lambda_0)(v) = 0$ and completes the proof that P_λ is a submersion.

We proceed by verifying that for every $a \in \mathbb{R}^n$ and for every m-dimensional linear subspace $W \subset \mathbb{R}^n$ we have $\ker D_xP_\lambda(a) \cap W = \{0\}$ for $\mathcal{L}^l$-almost all $\lambda \in \Lambda$.

Fix $\lambda_0 \in \Lambda$ such that $\ker D_xP_{\lambda_0}(a) \cap W = U$ with $\dim U = k$, where $1 \leq k \leq m$. Clearly, it is sufficient to prove that there is $\delta > 0$ such that $\ker D_xP_\lambda(a) \cap W = \{0\}$ for $\mathcal{L}^l$-almost all $\lambda \in B(\lambda_0, \delta)$. Let $e_1, \ldots, e_k$ be an orthonormal basis for U and let $M := \langle W \cup \ker D_xP_{\lambda_0}(a)\rangle$ be the subspace spanned by W and $\ker D_xP_{\lambda_0}(a)$. Observe that $k = \dim M^\perp$. For all $i = 1, \ldots, k$, consider the functions f^{e_i} defined above. Since P_λ is a submersion for all λ, we see that $\ker D_xP_\lambda(a)$ tends to $\ker D_xP_{\lambda_0}(a)$ as $\lambda \to \lambda_0$. Thus $Df^{e_i}(\lambda_0)([-1, 1]^n)$ is perpendicular to $\ker D_xP_{\lambda_0}(a)$ for all $i = 1, \ldots, k$. In particular, for each i there is a k-dimensional plane $W^{e_i} \subset V^{e_i}$ such that $Df^{e_i}(\lambda_0)(W^{e_i}) = M^\perp$. This implies the existence of $v \in \mathbb{R}^l$ such

that $Df^{e_1}(\lambda_0)v, \dots, Df^{e_k}(\lambda_0)v$ are linearly independent. Hence, for a sufficiently small $\varepsilon > 0$ we have $\ker D_x P_\lambda(a) \cap W = \{0\}$ for $\mathcal{L}^l$-almost all $\lambda \in B(\lambda_0, \varepsilon) \cap \langle v \rangle_{\lambda_0}$. By continuity, there exists $\delta > 0$ such that this is valid if we replace λ_0 by any $\lambda_1 \in B(\lambda_0, \delta)$. Finally, Fubini's theorem implies that $\ker D_x P_\lambda(a) \cap W = \{0\}$ for $\mathcal{L}^l$-almost all $\lambda \in B(\lambda_0, \delta)$.

The claim follows by choosing $W = T_a S$ and using the fact that since P_λ is a smooth submersion it is locally a fibration (see [GHL, Remark 1.92]). □

Now we are ready to prove the generalization of the Besicovitch-Federer projection theorem.

Proof of Theorem 1.2. The proof of the "only if"-part of Theorem 1.2 is similar to the one given in [Ma, p. 257-258]. Indeed, defining $E_{1,\delta}(\lambda)$ and $E_{2,\delta}(\lambda)$ as in Lemmas 2.2 and 2.3, setting

$$E_{3,\delta}(\lambda) := \{a \in E \mid (E \setminus \{a\}) \cap P_\lambda^{-1}(P_\lambda(a)) \cap B(a, \delta) \neq \emptyset\},$$

and applying Lemmas 2.2, 2.3 and 2.5, we conclude, as in [Ma, p. 257-258], that the claim holds.

To prove the "if"-part of the theorem, assume to the contrary that there is an m-rectifiable $F \subset E$ with $\mathcal{H}^m(F) > 0$. According to [F, Theorem 3.2.29], there exist m-dimensional C^1-submanifolds $S_1, S_2, \dots \subset \mathbb{R}^n$ such that $\mathcal{H}^m(F \setminus \cup_{i=1}^\infty S_i) = 0$.

Fixing i and letting a be a density point of $F \cap S_i$, Lemma 2.6 implies the existence of $\gamma > 0$ and $r > 0$ such that for $\mathcal{L}^l$-almost all $\lambda \in \Lambda$ we have $|P_\lambda(x) - P_\lambda(y)| \geq \gamma |x - y|$ for all $x, y \in B(a, r) \cap S_i$. This in turn gives that $\mathcal{H}^m(P_\lambda(E)) \geq \gamma^m \mathcal{H}^m(F \cap B(a, r) \cap S_i) > 0$ for $\mathcal{L}^l$-almost all $\lambda \in \Lambda$ which is a contradiction. □

Remark 2.7. In the "only if"-part of the previous proof we did not use the assumption that the mapping $(\lambda, x) \mapsto P_\lambda(x)$ is continuously differentiable in x (see Definition 2.4). It is sufficient to suppose that it is Lipschitz continuous. The differentiability in the second coordinate is needed only for the "if"-part of Theorem 1.2.

3. Dynamics of the geodesic flow

3.1. **Pairs of pants and right angle octagons.** The contents of this subsection and the following one are standard, see e.g. [Se]. Suppose S is a pair of pants with cuff lengths a, b and c (See Figure 1). The *seams* of S are the shortest geodesic segments connecting the cuffs. Consider the seam connecting the cuffs a and c and code by β and $\overline{\beta}$ the two sides of this seam. Analogously, consider the seam connecting the cuffs b and c and code by α and $\overline{\alpha}$ its two sides. If we cut S along these two seams, we obtain a hyperbolic octagon with right angles. We label the four sides of this octagon corresponding to the cut seams by the code of the part of S inside the octagon. The c cuff is cut into two geodesics of length $c/2$, which we label as c_1 and c_2. We see consecutively the labels α, b, $\overline{\alpha}$, c_1, $\overline{\beta}$, a, β and c_2 on the sides of the octagon (up to possibly exchanging the role of α and $\overline{\alpha}$, β and $\overline{\beta}$, or c_1 and c_2). Let R be a copy of the octagon inside the hyperbolic space $\mathbb{H}^2$.

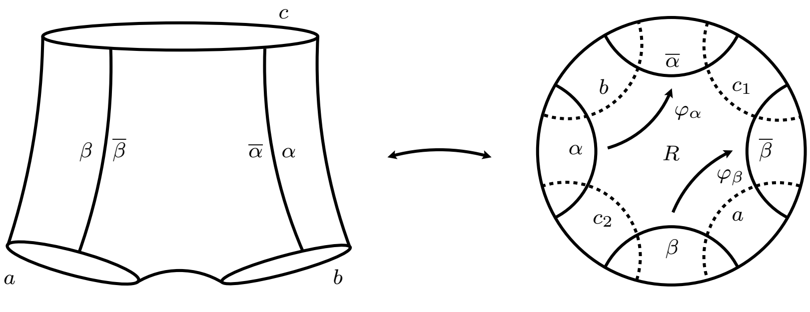

FIGURE 1. Pair of pants and the labelling of the sides of the octagon R.

For $\tau = \alpha$, $\overline{\alpha}$, β or $\overline{\beta}$, let φ_τ be the Möbius transformation sending the geodesic τ on the geodesic $\overline{\tau}$ (with the convention that $\overline{\overline{\tau}} = \tau$) and the half-plane separated by the complete extension of τ containing R onto the half-plane separated by the complete extension of $\overline{\tau}$ not containing R. We have $\varphi_{\overline{\tau}} = \varphi_\tau^{-1}$ for all τ. The union of S and its boundary ∂S is obtained from the closure of R by identifying the sides α and $\overline{\alpha}$ using φ_α and by identifying β and $\overline{\beta}$ using φ_β. Moreover, the geodesics extending the τ sides do not intersect one another, and therefore, by the classical ping-pong argument, φ_α and φ_β generate a free group G. The images of the interior of R by G are disjoint and the region containing R and delimited by the four extensions of the τ geodesics is a fundamental domain for G. For all $g \in G$, label the geodesic sides of gR by the image of the labelling of the geodesic sides of R. In a consistent way, each geodesic segment of the form $g\tau$ has two opposite labels corresponding to the two images of R that it separates.

We say that a geodesic γ in $T^1\mathbb{H}^2$ starts from R if $\gamma(0) \in \partial R$ and there is some $t > 0$ with $\gamma(t) \in R$. Let γ be a geodesic starting from R. It corresponds to a geodesic in $C(S)$ (recall the definition (1.1)), if and only if it never cuts the sides of $G(R)$ labelled as a, b, c_1 or c_2. In other words, γ intersects only τ geodesics. Record the interior label of these geodesics successively as ω_n, $n \in \mathbb{Z}$, ω_0 being the label of the side by which the geodesic γ enters R. This sequence is called the *cutting sequence* of γ. The cutting sequence of any geodesic in $C(S)$ is a reduced infinite word in α, $\overline{\alpha}$, β and $\overline{\beta}$, where reduced means that the succession $\tau\overline{\tau}$ is not permitted. Since two infinite geodesics in $\mathbb{H}^2$ with distinct supports are not at a bounded distance from each other, any cutting sequence is the cutting sequence of a unique geodesic. The boundary geodesics correspond to the reduced words $(\alpha)^\infty$, $(\overline{\alpha})^\infty$, $(\beta)^\infty$, $(\overline{\beta})^\infty$, $(\alpha\overline{\beta})^\infty$ and $(\overline{\alpha}\beta)^\infty$.

Consider the four disjoint complete geodesics in $\mathbb{H}^2$ extending the segments $\alpha, \overline{\alpha}, \beta$ and $\overline{\beta}$ of the previous subsection. Each of them cut S^1, the circle at infinity, into two intervals. Write A, $\overline{A}$, B and $\overline{B}$ for the interval separated from R by the geodesic α, $\overline{\alpha}$, β and $\overline{\beta}$, respectively. Let φ be defined on each T $(T = A, \overline{A}, B, \overline{B})$ by the corresponding Möbius transformation φ_τ. The mapping φ is expanding

(see [Se]) and $\varphi(T) = S^1 \setminus \operatorname{int} \overline{T}$, where the interior of a set in S^1 is denoted by int. In particular, $\varphi(T)$ contains the three intervals different from $\overline{T}$.

We define the boundary expansion of a point $\xi \in S^1$. If ξ does not belong to $\operatorname{int}(A \cup \overline{A} \cup B \cup \overline{B})$, stop here. Otherwise, let $\xi_0 = \alpha$, $\overline{\alpha}$, β or $\overline{\beta}$ accordingly. Apply then the procedure to $\varphi(\xi)$ and iterate. Every point has an empty, finite or infinite sequence of symbols attached, which is called its *boundary expansion.* Boundary expansions are reduced words in α, $\overline{\alpha}$, β and $\overline{\beta}$. The set of points with an infinite boundary expansion is a Cantor subset $\Omega \subset S^1$. For a geodesic γ starting in R, the positive part of the coding sequence is the boundary expansion of the limit point $\gamma(+\infty)$. Similarly, the sequence $\overline{\omega_0}, \overline{\omega_{-1}}, \overline{\omega_{-2}}, \ldots$ is the boundary expansion of $\gamma(-\infty)$. This defines a one-to-one correspondence Ψ between cutting sequences of geodesics starting from R and the set

$$(\Omega \times \Omega)^* = \{(\xi, \eta) \in \Omega \times \Omega \mid \xi_0 \neq \eta_0\},$$

namely, $\Psi(\omega) = (\xi, \eta)$ where $\xi_i = \omega_{i+1}$ and $\eta_j = \overline{\omega_{-j}}$ for $i, j = 0, 1, \ldots$

Clearly, if $\{\omega_n\}_{n\in\mathbb{Z}}$ is the cutting sequence of the geodesic γ, the shifted sequence $\{\omega'_n\}_{n\in\mathbb{Z}}, \omega'_n = \omega_{n+1}$ is associated to the geodesic $\gamma(\cdot + \ell)$, where ℓ is the first positive time t when $\gamma(t)$ is not in R. Consider the mapping

$$\Phi : \{(\xi, \eta, s) \mid (\xi, \eta) \in (\Omega \times \Omega)^*, 0 \leq s < \ell(\Psi^{-1}(\xi, \eta))\} \longrightarrow T^1\mathbb{H}^2$$

which associates to (ξ, η, s) the point $(x, v) \in T^1\mathbb{H}^2$ such that the geodesic γ with initial condition (x, v) satisfies $\gamma(+\infty) = \xi$, $\gamma(-\infty) = \eta$ and $\gamma(-s)$ is entering into R. The mapping Φ is a restriction of the usual chart of $T^1\mathbb{H}^2$ given by $(S^1 \times S^1)^* \times \mathbb{R}$. Its image is a subset of T^1R which is identified with $NW = D(S) \cup T^1(\partial S)$ (recall (1.2)). Metric properties of NW, and consequently those of $C(S)$, will be read from metric properties of Ω through this Lipschitz mapping Φ. Moreover, from the above symbolic representation, we see that NW is the nonwandering set of the geodesic flow on $T^1S \cup T^1(\partial S)$. The geodesic flow, restricted to $D(S) \cup T^1(\partial S)$, is therefore represented by a suspension over the set of reduced words with suspension function $\ell(\omega)$, where $\ell(\omega)$ is the time spent in R by the geodesic with cutting sequence ω.

3.2. **Markov repellers.** We use properties of Markov repellers as established by Bowen and Ruelle [R1]. A Markov repeller is an expanding piecewise $C^{1+\alpha}$ map of the real line into itself with a finite family of disjoint intervals $A_i, i \in J$, such that if $f(A_i)$ intersects the interior of some A_j, then $f(A_i)$ contains A_j. The set of points which remain in $\cup_{j\in J} A_j$ under applications of all the iterates f^n, $n \in \mathbb{N}$, is a Cantor set X. The set X is invariant under f. For any f-invariant probability measure μ on X consider the metric entropy $h_\mu(f)$. For any continuous function g on X, define the *pressure* $P(g)$ by

$$P(g) := \sup_\mu \left\{ h_\mu(f) + \int_X g \, d\mu \right\},$$

where μ varies over all f-invariant probability measures on X. Assume that f is topologically transitive. Then there exists a unique s with $0 < s < 1$ such that $P(-s \ln |f'|) = 0$. The number s is both the Hausdorff dimension and the packing dimension of X. More precisely, there exists a unique f-invariant probability measure μ_0 on X such that

$$h_{\mu_0} - s \int_X \ln |f'| \, d\mu_0 = 0.$$

The measure μ_0 is Ahlfors s-regular on X: for all ε small enough and for all $x \in X$ the ratio $\mu(B(x, \varepsilon))\varepsilon^{-s}$ is bounded away from 0 and infinity. In particular, $0 < \mathcal{H}^s(X) < \infty$.

Since the Patterson measure ν_0 is also Ahlfors regular [S1, Section 3], the measures ν_0 and μ_0 are mutually absolutely continuous with bounded densities. The geodesic flow invariant measure m constructed in [S1, Section 4] (called the Bowen-Margulis-Patterson-Sullivan measure) has support $D(S)$, is the measure of maximal entropy s for the geodesic flow on T^1S and has dimension $1 + 2s$.

Finally, if $(a, b, c) \mapsto f_{a,b,c}$ is a real analytic family of piecewise $C^{1+\alpha}$ expanding mappings, then the function $(a, b, c) \mapsto \dim_{\mathrm{H}}(X)$ is real analytic as well (see for example [R2, Corollary 7.10 and Section 7.28]).

3.3. **Proof of Theorem 1.1.** For fixed a, b and c, consider the set $\Omega_{a,b,c} \subset S^1$ of the previous subsection. It is a transitive Markov repeller for the mapping $\varphi_{a,b,c}$. The mapping $\varphi_{a,b,c}$ is given by a piecewise Möbius transformation, and therefore, it belongs to a semi-algebraic variety of piecewise analytic mappings. Moreover, $(a, b, c) \mapsto \varphi_{a,b,c}$ is real analytic, and thus the function $(a, b, c) \mapsto \delta(a, b, c) = \dim_{\mathrm{H}}(\Omega_{a,b,c})$ is real analytic. In particular, there is a two-dimensional submanifold of values a, b and c such that $\delta(a, b, c) = 1/2$.

Proposition 3.1. *Assume that $\delta(a, b, c) = 1/2$. Then the nonwandering set NW is purely 2-unrectifiable and has positive and finite 2-dimensional Hausdorff measure.*

Proof. It is enough to consider $D(S)$ since $T^1(\partial S)$ is 1-dimensional. Recalling that $\tau\overline{\tau}$ is a forbidden word for $\xi \in \Omega$, the above discussion implies that $D(S) = \cup_{i=1}^n U_i$ and each U_i is Lipschitz equivalent to an open subset of $\Omega \times \Omega \times I$, where I is a real interval. Since the measure μ_0 is Ahlfors $1/2$-regular on Ω, the measure $\mu_0 \times \mu_0 \times \mathcal{L}^1$ is Ahlfors 2-regular on $\Omega \times \Omega \times I$. Therefore $\dim_{\mathrm{H}}(\Omega \times \Omega \times I) = 2$ and $0 < \mathcal{H}^2(\Omega \times \Omega \times I) < \infty$. Thus $\dim_{\mathrm{H}}(D(S)) = 2$ and $0 < \mathcal{H}^2(D(S)) < \infty$. For the first claim it is enough to notice that $\Omega \times \Omega$ is purely 1-unrectifiable, since it is a product of two Cantor sets of dimension $1/2$ [Ma, Example 15.2]. Thus the product $\Omega \times \Omega \times I$ is purely 2-unrectifiable, and so is $D(S)$. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. In [JJL, Section 3] it is shown that locally there exist an open set $U \subset T^1S$, bi-Lipschitz mappings $\psi_1 : U \to I^3$ and $\psi_2 : I^2 \to \Pi(U)$ and a

smooth mapping $P : I^3 \to I^2$ such that

$$\Pi|_U = \psi_2 \circ P \circ \psi_1,$$

where $I \subset \mathbb{R}$ is the open unit interval. The mapping P is defined by $P(y_1, y_2, t) = (P_t(y_1, y_2), t)$, where $\{P_t : I^2 \to I\}_{t \in I}$ is a transversal family of smooth mappings.

By Proposition 3.1, the set $\psi_1(D(S) \cap U) = E \times I$ is purely 2-unrectifiable. Thus $E \subset I^2$ is purely 1-unrectifiable. Furthermore, $P(E \times I) = \bigcup_{t \in I} P_t(E) \times \{t\}$. By Theorem 1.2,

$$\mathcal{H}^1(P_t(E)) = 0 \text{ for } \mathcal{L}^1\text{-almost all } t \in I,$$

giving $\mathcal{H}^2(P(E \times I)) = 0$ by Fubini's theorem. This implies that

$$\begin{aligned} \mathcal{H}^2(\Pi(D(S) \cap U)) &= \mathcal{H}^2((\psi_2 \circ P \circ \psi_1)(D(S) \cap U)) \\ &= \mathcal{H}^2(\psi_2(P(E \times I))) = 0, \end{aligned}$$

since ψ_2 is a bi-Lipschitz mapping. The claim follows from the fact that $T^1 S$ can be covered by countably many open sets U. □

Corollary 3.2. *The Bowen-Margulis-Patterson-Sullivan measure m is 2-dimensional, its support spt $m = D(S)$ is 2-dimensional and $\mathcal{L}^2$(spt $(\Pi_* m)) = 0$.*

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